Lecture I: Parametric Oscillators

Key Idea and Example:

- Oscillator with sinusoidally varying frequency
- Mathieu Equation

Contents:

- Ideas, trends
- Proof of nature of solutions
- Variation of parameters calculation for growth rate, threshold

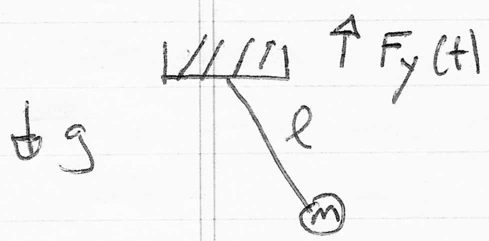
Result:

- Parametric growth calculation

w) Parametric Instability

→ consider pendulum with support acted on by vertical force

$\downarrow + \Rightarrow$ down



so $g \Rightarrow g - F_y(t)/m$

$$\therefore 0 = \ddot{\theta} + \frac{g}{\ell}\theta \rightarrow \ddot{\theta} + \left(\frac{g}{\ell} - \frac{a(t)}{\ell}\right)\theta = 0$$

let $a(t) = a_0 \cos(\alpha t)$

$$\Rightarrow \ddot{\theta} + \omega_0^2 \theta - \frac{a_0}{\ell}\cos(\alpha t)\,\theta = 0$$

$\left\{ \begin{array}{l} \text{see small } \phi \text{ limit, pg. 12} \\ \alpha \not\gg \omega_0 \end{array} \right.$

$\therefore$ of Mathieu's equation genre, i.e.

$$\left[\ \ddot{x} + \omega_0^2(1 + a\cos(\gamma t))\,x = 0\right.$$

$\Downarrow$

$\omega^2 = \omega^2(t)$, hence parametric oscillator

Parametric oscillator $\leftrightarrow$ $\omega^2(t)$ periodic
oscillation of effective frequency

→ Some observations :

a) informal – consider what might happen?

for instability, observe can produce secularity if $\gamma \sim 2\omega_0$, via beat at fundamental

$$\underbrace{\ddot{x} + \omega_0^2 x} + a \underbrace{\cos(\gamma t)}_{\downarrow\ \cos(2\omega_0 t)} \omega_0^2 \underbrace{x}_{\downarrow\ \sin(\omega_0 t)} = 0$$

$\sim \sin(\omega_0 t)$

resonant drive of fundamental oscillator

$\Rightarrow$ secularity $\rightarrow$ instability (why?)

$\therefore$ Solution of oscillator at ω_0 beats with parameter oscillation $\Rightarrow$ secularity

$\therefore$ parametric resonance at/near $\gamma \sim 2\omega_0$ (twice fundamental)

Note: here $\omega^2 = \omega^2(t) \Rightarrow \partial L/\partial t \neq 0$ energy not conserved

$\Rightarrow$ work done on system (e.g. LGM oscillating pendulum support)

$\Rightarrow$ source of energy for instability

? What is relation of this to 3-mode parametric instability calculation (2004) ?

b) formal (Floquet theory) $\Bigg\{$ what Mathematics Predicts

$\Rightarrow$ (What type solution possible!)

- $\omega(t)$ periodic, with period $T = 2\pi/\gamma$

$\therefore$ $\begin{cases} \omega(t+T) = \omega(t) \\ \text{eqn. invariant under } t \to t+T \end{cases}$

$\therefore$ if $x_1(t)$, $x_2(t)$ are 2 independent solutions of basic eqn.

$\Rightarrow$ $x_1(t)$, $x_2(t)$ must transform to linear combinations of themself upon $t \to t+T$ (linear eqn!)

and

can choose x_1, x_2 s/t

$$\begin{cases} x_1(t+T) = \mu_1 x_1(t) \\ x_2(t+T) = \mu_2 x_2(t) \end{cases}$$

(here "can choose" means can diagonalize transformation matrix)

$\Rightarrow$ most general functions having this property are:

$$\begin{cases} x_1(t) = \mu_1^{t/T}\,\Pi_1(t) \\ x_2(t) = \mu_2^{t/T}\,\Pi_2(t) \end{cases}$$

$\Bigg\{$ where: $\Pi_{1,2}(t+T) = \Pi(t)$

- second, observe since linear equation
  $\Rightarrow$ Wronskian constant

$$\dot{x}_2 x_1 - \dot{x}_1 x_2 = \text{const.}$$

$$\begin{aligned} x_2 &\left(\ddot{x}_1 + \omega^2(t)\, x_1\right) = 0 \\ x_1 &\left(\ddot{x}_2 + \omega^2(t)\, x_2\right) = 0 \end{aligned} \qquad \Rightarrow \frac{d}{dt}\left(x_2 \dot{x}_1 - \dot{x}_2 x_1\right) = 0$$

but $\begin{cases} W(x_1, x_2) = (\mu_1 \mu_2)^{-1}\, W\big(x_1(t+T), x_2(t+T)\big) \\ \text{i.e. consider time translation by } T \end{cases}$

$\rightarrow \therefore$ $\boxed{\mu_1 \mu_2 = 1}$

$$W(x_1, x_2) = \left(\mu_2^{t/T}\, \Pi_2\right)\left(\mu_1^{t/T}\, \Pi_1\right) - \left(\mu_1^{t/T}\, \Pi_1\right)\left(\mu_2^{t/T}\, \Pi_2\right)$$

$$= \left(e^{(\ln \mu_2)\, t/T}\, \Pi_2\right) e^{(\ln \mu_1)\, t/T}\, \Pi_1 - \left(e^{(\ln \mu_1)\, t/T}\, \Pi_1\right) e^{\ln \mu_2\, t/T}\, \Pi_2$$

- Can also observe:
  1) coeffs in oscillator real, so
     $x(t)$ an integral $\rightarrow x^*$ a solution
  $\Rightarrow$
  2) μ_1, μ_2 same as μ_1^*, μ_2^*
     i.e.

$$\underset{\text{I}}{\begin{aligned} \mu_1 &= \mu_2^* \\ \mu_2 &= \mu_1^* \end{aligned}} \quad \text{or} \quad \underset{\text{II}}{\begin{aligned} \mu_1 &= \mu_1^* \\ \mu_2 &= \mu_2^* \end{aligned}} \Big\rangle \text{ both real} \quad \ldots$$

if I, $\mu_1 \mu_2 = 1 \Rightarrow \begin{aligned} \mu_1 &= 1/\mu_1^* \\ \mu_2 &= 1/\mu_2^* \end{aligned} \Rightarrow \underline{|\mu_1|^2 = |\mu_2|^2 = 1}$

(trivial)

if (II) $\mu_1 \mu_2 = 1$; μ_1, μ_2 real $\Rightarrow$

$\Rightarrow$ $x_1(t) = \mu^{t/T}\Pi_1(t)$, $x_2(t) = \mu^{-t/T}\Pi_2(t)$

i.e. 1 increasing, 1 decreasing } solution $\Rightarrow$ { parametric instability

[N.B. Exponential, not secular, growth] !

$\Rightarrow$ "true" instability is possible

$\rightarrow$ Some Calculation (as basic structure of the solution established).

Consider Mathieu's eqn:

$$\ddot{x} + \omega_0^2\left[1 + h\cos[(2\omega_0 + \epsilon)t]\right]x = 0$$

bounds on ϵ for instability ?

For solution, SHO $\Rightarrow$

$$x = a\cos(\omega_0 t) + b\sin(\omega_0 t)$$

so, in spirit of multiple-time-scale P.T. (i.e. $\omega^2(t)$ enters via $h \ll 1$ $\Rightarrow$ expect slow time scale variation of coefficients)

mis-match vs h

$$x = a(t)\cos[(\omega_0 + \epsilon/2)t] + b(t)\sin[(\omega_0 + \epsilon/2)t]$$

coeffs become slowly varying

Plugging it in:

$$\ddot{x} = \ddot{\left(a(t)\cos[(\omega_0+\epsilon/2)t]\right)} + \text{O.T.}$$

$\hookrightarrow$ other term

$$= -(\omega_0+\epsilon/2)^2 a(t)\cos[\,] - 2(\omega_0+\epsilon/2)\dot{a}(t)\sin[\,] + \ddot{a}\cos[\,] + \text{O.T.}$$

neglect $\ddot{a}$, $\ddot{b}$ as h.o. in slowness (recall amplitude eqn. deriv.)
and

$\Rightarrow$ ω_0^2 term, only

$$-(\omega_0+\epsilon/2)^2 a(t)\cos[\,] - 2\dot{a}(t)(\omega+\epsilon/2)\sin[\,]$$

$$-(\omega_0+\epsilon/2)^2 b(t)\sin[\,] + 2\dot{b}(t)(\omega+\epsilon/2)\cos[\,]$$

ω_0^2

$$+\,\omega_0^2\left[\cancel{a(t)}\cos[\,] + \cancel{b(t)}\sin[\,]\right]$$

$$+\,\omega_0^2 h\cos((2\omega_0+\epsilon)t)\left[a(t)\cos[\,] + b(t)\sin[\,]\right]$$

$$= 0$$

Now; – neglect $O(\epsilon^2)$ terms $\Rightarrow$ only $\omega_0\epsilon$ term survives.

– observe $\cos\left[(\omega_0+\frac{\epsilon}{2})t\right]\cos\left[(2\omega_0+\epsilon)t\right]$

$$= \frac{1}{2}\cos\left[3(\omega_0+\epsilon/2)t\right] + \frac{1}{2}\cos\left[(\omega_0+\epsilon/2)t\right]$$

{fast oscillation / non-resonant} $\rightarrow$ (i.e. expect h.o. in h)

Resonant with fundamental

Resonant contribution is interesting one here

$\Rightarrow$

$$-\omega_0\epsilon\big(a(t)\cos[\,] + b(t)\sin[\,]\big)$$
$$-2\dot{a}(\omega_0+\epsilon/2)\sin[\,] + 2\dot{b}(\omega_0+\epsilon/2)\cos[\,]$$
$$+\frac{\omega_0^2}{2}h\Big[a(t)\cos[\,] - b(t)\sin[\,]\Big]$$

$$=0$$

Regrouping coeffs. $\cos[\,]$, $\sin[\,]$;

$$\sin[\,]\left(-2\omega_0\dot{a} - b\omega_0\epsilon - \omega_0^2 h\, b/2\right)$$
$$+\cos[\,]\left(2\dot{b}\omega_0 - a\epsilon\omega_0 + \tfrac{1}{2}h\omega_0^2 a\right) = 0$$

$\Rightarrow$

$$(2\omega_0)\dot{a} + (\omega_0\epsilon)b + \left(\frac{\omega_0^2}{2}h\right)b = 0$$
$$(2\omega_0)\dot{b} - (\omega_0\epsilon)a + \left(\frac{\omega_0^2 h}{2}\right)a = 0$$

$\Rightarrow$

$$\boxed{\begin{cases}\dot{a} + (\epsilon/2)b + (\omega_0 h/4)b = 0\\ \dot{b} - (\epsilon/2)a + (\omega_0 h/4)a = 0\end{cases}}$$

Basic system of Eqns for Amplitude Variation

$a(t) = a_0 e^{st}$
$b(t) = b_0 e^{st}$

exponentially growing/damping solution

$\Rightarrow$

$$s\, a_0 + \left(\epsilon/2 + \omega_0 h/4\right) b_0 = 0$$

$$\left(-\frac{\epsilon}{2} + \frac{\omega_0 h}{4}\right) a_0 + s\, b_0 = 0$$

$$\therefore \quad \boxed{s^2 = \frac{\omega_0^2 h^2}{16} - \frac{\epsilon^2}{4} = \frac{1}{4}\left(\frac{\omega_0^2 h^2}{4} - \epsilon^2\right)}$$

$\Rightarrow$ Parametric instability criterion.

Growth rate

Observe:

- instability for:

$$\epsilon^2 = (\gamma - 2\omega_0)^2 < \frac{\omega_0^2 h^2}{4}$$

$\omega_0 \rightarrow$ fundamental
$\gamma \rightarrow$ parametric variation freq.

amplitude of variation

$h^2 > 4(\gamma - \omega_0)^2/\omega_0^2$

i.e. sufficiently close to resonance $\Rightarrow$ growth.

for $(\gamma - 2\omega_0)^2 > \omega_0^2 h^2/4 \rightarrow$ oscillation

- amplitude of $\omega_0^2(t)$ variation sets proximity threshold

more generally, can show when $\overset{\text{integer}}{\downarrow}$ $n\gamma = 2\omega_0$
$\Rightarrow$ parametric resonance. Of course, higher
$n \Rightarrow$ resonance region $\sim h^n$

- with friction, find threshold for instability:

i.e. $(\gamma - 2\omega_0)^2 < \left[\left(\frac{4}{2} h \omega_0\right)^2 - 4\alpha^2\right]$

friction coeff. (for α)

i.e. P.I. growth must beat damping!.
Friction raises required h.

- Pumping on swing

$\rightarrow$ "pumping" $\rightarrow$ change of I

$$\ddot{\theta} + \frac{mgl\,\theta}{I(t)} = 0$$

$$I(t) = I_0 + \epsilon I_1(t)$$

$$\ddot{\theta} + \frac{g}{l}\theta + \frac{\epsilon g}{l}\frac{\Delta I(t)}{I}\theta = 0$$
$$+ \alpha\dot{\theta}$$

need pump twice per cycle